CALCULATION OF THE NONSTATIONARY THERMOELASTIC STRESSES IN A PLANE WALL

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An approximate method is proposed for calculating the nonstationary thermoelastic stresses in a plane wall for an arbitrary law of variation of fluid temperature with time.

The calculation of the thermal stresses in the structural elements of power plants can often be reduced to calculation of the stresses in a plane wall rigidly fixed along the edges. Usually it is possible to neglect the dynamic component of the stresses, assuming that the stress field is quasi-stationary. In this case if the temperature field of the element is known, in order to find the thermoelastic stresses it is possible to use the results obtained in [1].

The most dangerous thermal stresses usually develop at the surface of the element. Therefore in what follows, in speaking of the stresses, we will have in mind the tangential component of the stresses at the surface of a wall washed by a fluid.

In [2] Bagdasarov derived formulas for calculating the thermoelastic stresses in plane structural elements for different laws of variation of the temperature of the heat transfer agent with time with and without thermal shielding. Calculations based on [2] are rather clumsy owing to the need to evaluate four to six terms of the series.

In engineering practice it is also useful to have an approximate method of calculating stresses which, while simple, has sufficient accuracy. Such a method is proposed below. It is suitable for cases with any laws of variation of the temperature of the heat transfer agent with time and for plates with and without thermal shielding.

The solution of the problem is based on the "parabolic approximation" [4]. The process of heat transfer to an infinite plane wall of thickness δ thermally insulated on one side and washed by a fluid on the other can be described by the following system:

$$\frac{\partial t}{\partial \tau} = a\frac{\partial^2 t}{\partial x^2},$$

$$x = 0, \quad \frac{\partial t}{\partial x} = 0;$$

$$x = \delta, \quad -\lambda\frac{\partial t}{\partial x} = \alpha(t - \Theta).$$

Using the expression for the average temperature of the plate

$$\bar{t} = \frac{1}{\delta}\int_0^\delta t(x)\,dx,$$

and introducing the notation

$$k(\tau) = \left.\frac{\partial t}{\partial x}\right|_{x=\delta}\cdot \delta/[t(\delta) - \bar{t}],$$

we can replace this system with the single equation

$$\frac{d\bar{t}}{d\tau}\delta c\gamma = (\Theta - \bar{t})\alpha_*, \tag{1}$$

where

$$\alpha_* = \frac{1}{1/\alpha + \delta/k(\tau)\lambda}; \tag{2}$$

$k(\tau)$ is a function of time which varies within rather narrow limits for any values of the Bi number. If we approximate the temperature field of the plate with a parabola [4] $t(x) = t(x = \delta) + \Delta t(x/\delta)^2$, then $k = 3$ and does not depend on time. Accordingly, we can rewrite Eq. (1) as follows:

$$\frac{d\bar{t}}{d\,\mathrm{Fo}} + \varphi\bar{t} = \varphi\Theta, \tag{3}$$

$$\varphi = 3\,\mathrm{Bi}/(3 + \mathrm{Bi}). \tag{4}$$

Solving Eq. (3) for the initial condition $\bar{t}(0) = \Theta(0) = t_0$, we obtain

$$\bar{t} = \left[t_0 + \int_0^{\mathrm{Fo}} \varphi\Theta\exp(\varphi\,\mathrm{Fo})d\,\mathrm{Fo}\right]\exp(-\varphi\,\mathrm{Fo}). \tag{5}$$

We find the surface temperature of the plate from the condition

$$t_s - \Theta = -\frac{1}{\mathrm{Bi}}\frac{d\bar{t}}{d\,\mathrm{Fo}}. \tag{6}$$

Hence

$$t_s = \frac{\varphi t_0}{\mathrm{Bi}} + \frac{\varphi\Theta}{3} + \frac{\varphi^2}{\mathrm{Bi}}\exp(-\varphi\,\mathrm{Fo})\times \int_0^{\mathrm{Fo}}(\Theta - t_0)\exp(\varphi\,\mathrm{Fo})\,d\,\mathrm{Fo}.$$

On the basis of [1] the thermoelastic stresses at the surface of the plate

$$\sigma = \frac{\alpha_T E}{1-\nu}(\bar{t} - t_s).$$

Therefore

$$\sigma = \left[(t_0 - \Theta)\frac{\varphi}{3} + \frac{\varphi^2}{3}\exp(-\varphi\,\mathrm{Fo})\times \int_0^{\mathrm{Fo}}(\Theta - t_0)\exp(\varphi\,\mathrm{Fo})\,d\,\mathrm{Fo}\right]\frac{\alpha_T E}{1-\nu}. \tag{7}$$

We will obtain formulas for certain laws of variation of the temperature of the heat transfer agent with time.

Exponential law:

$$\Theta = t_0 - \Delta\Theta[1 - \exp(-\mathrm{PdFo})] = t_0 - \Delta\Theta[1 - \exp(-m\tau)].$$

After substitution in (7) and integration we obtain

$$\bar{\sigma} = \frac{\mathrm{Pd}\,\varphi}{3(\varphi - \mathrm{Pd})}\left[\exp(-\mathrm{Pd}\,\mathrm{Fo}) - \exp(-\varphi\,\mathrm{Fo})\right],$$

$$\bar{\sigma} = \sigma(1-\nu)/\Delta\Theta\, E\,\alpha_T. \qquad (8)$$

After differentiation with respect to Fo and determination of the maximum, we find

$$\bar{\sigma}_{max} = \frac{\varphi}{3}\left(\frac{\varphi}{\mathrm{Pd}}\right)^{-\varphi/(\varphi-\mathrm{Pd})}. \qquad (9)$$

The stress maximum corresponds to the dimensionless time

$$\mathrm{Fo}_* = (\ln\varphi - \ln \mathrm{Pd})/(\varphi - \mathrm{Pd}). \qquad (10)$$

If $\mathrm{Pd} \to \infty$ (instantaneous jump in fluid temperature), then

$$\bar{\sigma}_{max} = \varphi/3 = \mathrm{Bi}/(3+\mathrm{Bi}). \qquad (11)$$

The maximum error of this formula lies in the region Bi = 6–8 and does not exceed 30% (on the high side). As will be shown below, for other values of Bi and Pd the error of calculation based on (9) and (11) is much smaller. In the region Bi < 6 expression (11) coincides in form with Manson's formula [5] obtained for the same conditions by approximation of the exact formulas, but differs somewhat in relation to the coefficients. The advantages of formulas (9) and (11) are their simplicity and the possibility of using them for any values of the Bi and Pd numbers.

In order to illustrate the satisfactory accuracy of the solution obtained within the ranges of Bi and Pd of practical importance, we have calculated the stresses for three v alues of Bi : 1,4, ∞ and for Pd varying from 0 to 20. The results are presented in Fig. 1. The same figure shows the corresponding $\bar{\sigma}_{max} = f(\mathrm{Bi}, \mathrm{Pd})$ curves calculated from the exact formulas.

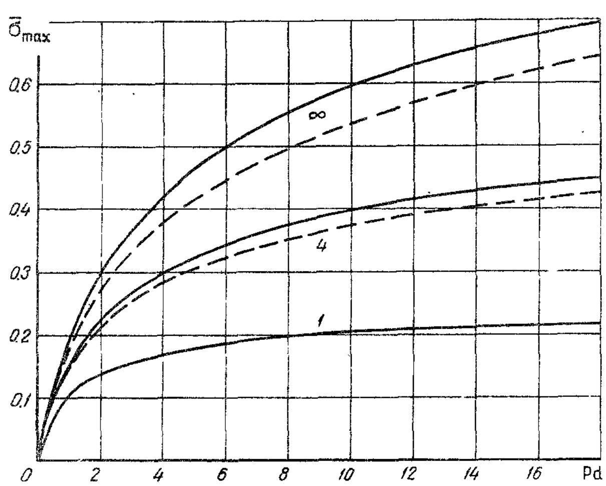

Fig. 1. Maximum value of thermoelastic stresses. Solid curves from Eq. (9), dashed curves from exact formulas. The figures on the curves are values of the Bi number.

As a comparison shows, the accuracy of the calculations is quite satisfactory and is greater, the smaller the Bi number. Even at Bi = 4 the exact and approximate $\sigma_{max} = f(\mathrm{Pd})$ curves differ only slightly, while at Bi = 1 they merge. The maximum error at Bi = ∞ is about 10%, the stresses found from the approximate formula being higher than those found from the exact formula. Therefore the error is on the safe side.

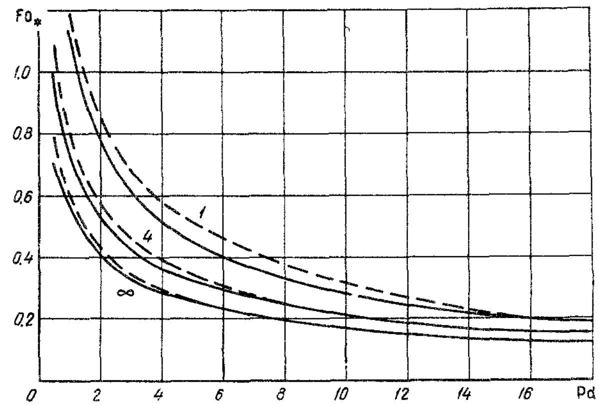

Fig. 2. Time corresponding to stress maximum as a function of Pd number. Solid curves from Eq. (10), dashed curves from exact formula. The figures on the curves are values of the Bi number.

The error in determining Fo_* is also small. Figure 2 presents $\mathrm{Fo}_* = f(\mathrm{Bi}, \mathrm{Pd})$ curves found from the exact and approximate formulas for the same values of Bi. A comparison of the corresponding curves shows that the accuracy of the calculation is satisfactory.

As may be seen from Eqs. (1), (3), the parabolic approximation is based on the laws of the regular thermal regime except that in the regular regime the thermal resistance of the cooled or heated body is constant in time but depends on the Bi number, whereas, in accordance with the parabolic approximation, the thermal resistance is constant in time and does not depend on Bi.

For boundary conditions of the third kind the variation of the temperature field of the plate with time is described in the regular regime by an exponential whose exponent $(-\mu^2)$ is determined from the equation [6]

$$\mathrm{ctg}\,\mu = \mu/\mathrm{Bi}.$$

In the parabolic approximation the cooling rate is equal to $(-\varphi)$. From the condition $\varphi = \mu^2$ we can obtain the formula

$$k = \frac{1}{1/\mu^2 - 1/\mathrm{Bi}}. \qquad (12)$$

The coefficient k calculated from this formula varies monotonically from 3 (at Bi = 0) to 2.47 (at Bi = ∞). Above, we set k = 3 for all Bi. Therefore the improvement in accuracy as Bi decreases is understandable. It is also explained by the fact that as Bi decreases so does the effect of the thermal resistance of the plate on the transient process, while the role of the thermal resistance to heat transfer increases. In this case the various assumptions regarding the thermal resistance of the plate have less influence on the accuracy of the calculations. At large values of Bi the initial interval of disordered heat transfer has a strong influence on the entire transient process. This explains why an attempt to improve the accuracy of the calculations at large Bi by introducing k = f(Bi) in accordance with Eq. (12) was unsuccessful. It is expedient to take k = 3 irrespective of Bi.

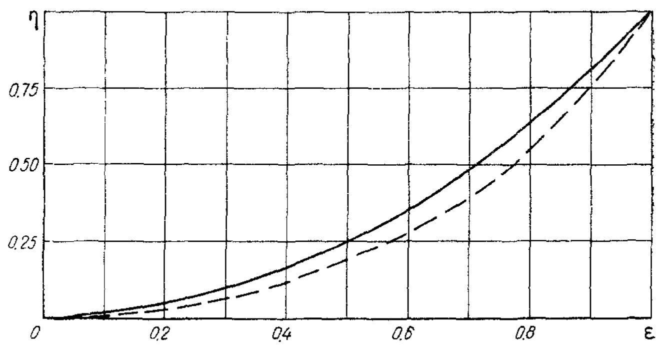

Fig. 3. η as a function of ε and the Pd number for Bi = ∞. The solid line represents the curves for Pd = 2 and 4 and the curve for ε^2 (in the graph all three curves merge into one). The dashed line represents the relation for Pd = 100. The relations for all Pd were calculated from the exact formula.

Linear law:

$$\Theta = t_0 - \Delta\Theta\,\mathrm{Fo},$$

$$\bar{\sigma} = \frac{\sigma(1-\nu)}{\Delta\Theta\alpha_T E} = [1-\exp(-\varphi\,\mathrm{Fo})]\cdot\frac{1}{3}. \qquad (13)$$

In the quasi-stationary regime as Fo → ∞ stresses $\bar{\sigma}$ = 1/3 develop in the plate, which coincides with the exact solution.

If the temperature of the heat transfer agent varies linearly by an amount ΔΘ ΔFo in time ΔFo, after which it remains constant, then for Fo ≥ ΔFo

$$\bar{\sigma} = \frac{1}{3}[\exp(\varphi\Delta\,\mathrm{Fo}) - 1]\exp(-\varphi\,\mathrm{Fo}). \qquad (14)$$

Obviously, the stress maximum occurs at the moment ΔFo and is equal to

$$\bar{\sigma}_{max} = \frac{1}{3}[1-\exp(-\varphi\Delta\,\mathrm{Fo})]. \qquad (15)$$

When the quantity ΔFo tends to infinity, we arrive at the previous results. The results of calculations based on these formulas are in good agreement with Fritz's nomogram [3].

The parabolic approximation can also be used to find the thermoelastic stresses in a plate in the presence of thermal shielding. We will consider a plate of thickness δ composed of a structural wall of thickness R and thermal shielding of the same material washed by a heat transfer agent with varying temperature. To be on the safe side, we assume ideal thermal contact between the structural wall and the thermal shielding. On the basis of the parabolic approximation the temperature field in the plate is described by the equation

$$t(x) = t(0) - [t(\delta) - \Theta]\frac{\mathrm{Bi}}{2}\left(\frac{x}{\delta}\right)^2.$$

The average temperature of the structural wall

$$\bar{t}_w = t(0) - [t(\delta) - \Theta]\frac{\mathrm{Bi}}{6}\varepsilon^2.$$

The temperature of the wall surface

$$t_{ws} = t(0) - [t(\delta) - \Theta]\frac{\mathrm{Bi}}{2}\varepsilon^2.$$

Hence the thermal stresses in the wall

$$\bar{\sigma}_w = \frac{E\alpha_T}{1-\nu}[t(\delta) - \Theta]\,\mathrm{Bi}\frac{\varepsilon^2}{3}. \qquad (16)$$

But the quantity

$$\frac{E\alpha_T}{1-\nu}[t(\delta) - \Theta]\frac{\mathrm{Bi}}{3} = \bar{\sigma}$$

represents the stresses at the surface of a monolithic plate of thickness δ. The ratio of the stresses in the structural wall to the stresses in a plate whose thickness is equal to the total thickness of the wall and the thermal shield

$$\eta = \sigma_w/\sigma = \varepsilon^2. \qquad (17)$$

Thus, in the parabolic approximation the quantity η depends only on ε. In the exact formulation η also depends on Bi and Pd, but the effect of these criteria is relatively small and may be neglected.

In order to demonstrate this, we calculated the relations between η and ε for Bi = ∞ and three values of Pd: 2, 4, and 100 from the exact formulas of [2]. The value Bi = ∞ was selected because, as is clear from the foregoing, at that value the maximum deviation of the approximate from the exact results is to be expected. The calculations were made for an exponential law of variation of the temperature of the heat transfer agent with time. The results are presented in Fig. 3. The same figure shows the curve $\eta = \varepsilon^2$. Clearly, the exact curve $\eta = f(\varepsilon)$ for Pd = 100 differs somewhat from the ε^2 curve, passing below the latter; the exact curves for Pd = 2 and Pd = 4 coincide completely with the $\eta = \varepsilon^2$ curve. A certain exaggeration of the result at large values of Pd is on the safe side.

Thus, it has been shown that the approximate method based on the parabolic approximation is suitable for engineering calculations of the thermoelastic stresses in a plane wall with and without thermal shielding for an arbitrary law of variation of the temperature of the heat transfer agent with time.

NOTATION

σ represents stresses; τ is time; x is the coordinate in direction normal to the surface of the plate; t is the temperature of plate; $\bar{t}$ is the average temperature of plate; t_s is the surface temperature; Θ is the temperature of fluid; δ is the thickness of plate; R is the thickness of structural wall; E is the modulus of elasticity; ν is Poisson's ratio; α_T is the coefficient of linear expansion; Bi = $\alpha\delta/\lambda$; Fo = $a\tau/\delta^2$. For an exponential law of variation of the fluid temperature with time Pd = $m\delta^2/a$; $\varepsilon = R/\delta$.

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